SHARP ISOPERIMETRIC UPPER BOUNDS FOR PLANAR STEKLOV EIGENVALUES

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ABSTRACT. We solve the isoperimetric problem for the first and second nonzero Steklov eigenvalues of planar domains, without any assumption on the number of connected components of the boundary. Our approach uses the known sharp upper bounds for the weighted Neumann eigenvalues, and a homogenisation method allowing to approximate these eigenvalues by the Steklov eigenvalues of appropriately chosen perforated subdomains.

1. INTRODUCTION

For a connected bounded open set $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$, the Steklov problem consists in determining all $\sigma \in \mathbb{R}$ for which the following boundary problem admits a nontrivial solution:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \partial_n u = \sigma u & \text{on } \partial\Omega. \end{cases}$$

The eigenvalues of this problem form a sequence $0 = \sigma_0 < \sigma_1(\Omega) \leqslant \sigma_2(\Omega) \leqslant \ldots \nearrow \infty$, where each eigenvalue is repeated according to its multiplicity. For each $k \in \mathbb{N}$, we investigate sharp upper bounds for $\sigma_k(\Omega)$; to that end, define the scale invariant quantities

$$\Sigma_k(\Omega) := \sigma_k(\Omega)\,|\partial\Omega|^{\frac{1}{d-1}}.$$

It follows from Colbois–El Soufi–Girouard [5] that $\Sigma_k(\Omega) < Ck^{2/d}$ for some dimensional constant C, and it is therefore meaningful to study the extremal eigenvalues for $\Sigma_k(\Omega)$:

$$\Sigma^*_{k,d} := \sup_{\Omega\subset\mathbb{R}^d} \Sigma_k(\Omega).$$

The question to determine these was raised in [11, Open problem 2], we answer it for $d = 2$, $k \in \{1,2\}$, and obtain a lower bound on the optimal upper bound for $d = 2$, $k \geqslant 3$.

1.1. **Isoperimetric bounds in the plane.** Attempts at maximising Steklov eigenvalues normalised by perimeter go back to the work of Weinstock [25]. He proved in 1954 that for simply-connected planar domains, $\Sigma_1(\Omega) \leq 2\pi$, with equality if and only if Ω is a disk. This was followed by works of Hersch–Payne–Schiffer [14], then later Girouard–Polterovich [10] and Karpukhin [15] who proved that

$$\Sigma_k(\Omega) \leq 2\pi(k + \gamma + b - 1),$$

this time for compact surfaces Ω of genus γ with b boundary components. It follows from Girouard–Polterovich [26] that for $\gamma = 0$ and $b = 1$, this bound is saturated by a sequence of simply-connected domains $\Omega^\varepsilon \subset \mathbb{R}^2$ that degenerates to a union of k identical disks as $\varepsilon \to 0$. Bounds for Σ_k which do not depend on the number of boundary components are notably more difficult to obtain. For Ω a compact orientable surface of genus 0 with boundary, it was proved by Kokarev [18] that

$$\Sigma_1(\Omega) < 8\pi, \tag{1}$$

and it follows from the work of Karpukhin–Stern [17] that

$$\Sigma_2(\Omega) < 16\pi. \tag{2}$$

It follows from Girouard–Lagacé [9] that these inequalities are sharp and saturated by sequences of domains in the sphere $\mathbb{S}^2$ equipped with appropriate Riemannian metrics. In the present paper, we prove that both inequality (1) and inequality (2) remain sharp for planar domains.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply-connected domain with Lipschitz boundary. There exists a sequence $\Omega^\varepsilon \subset \Omega$ of subdomains, with $\partial\Omega \subset \partial\Omega^\varepsilon$, such that*

$$\sigma_k(\Omega^\varepsilon)|\partial\Omega^\varepsilon| \xrightarrow{\varepsilon\to 0} 8\pi k.$$

Together with (1) and (2), Theorem 1.1 leads to the following

Corollary 1.2. *The extremal eigenvalues for planar domains $\Sigma_k^* := \Sigma_{k,2}^*$ satisfy*

$$\Sigma_1^* = 8\pi, \qquad \Sigma_2^* = 16\pi, \qquad \Sigma_k^* \geqslant 8\pi k.$$

Following [7, Theorem 4.3], it was suggested in [11] that the number of boundary components in a maximizing sequence for Σ_1 needs to be unbounded. Inequality (1), together with the monotonicity results of [7, Theorem 4.3] and [21, Theorem 1.3] yield the following corollary, confirming this claim.

Corollary 1.3. *Any sequence of domains such that $\Sigma_1(\Omega) \to 8\pi$ has a number of boundary components going to $+\infty$.*

The proof of Theorem 1.1 uses homogenisation theory to approximate eigenvalues of an inhomogeneous Neumann problem for an appropriate weight $\beta > 0$ that is related to a well chosen Riemannian metric on the sphere $\mathbb{S}^2$. We are naturally led to the following conjecture.

Conjecture. *For each $k \in \mathbb{N}$,*

$$\Sigma_k^* = 8\pi k.$$

See [9] for a more general conjecture which, together with Theorem 1.1 would lead to this one.

1.2. **Flexibility in the prescription of the Steklov spectrum.** Bucur–Nahon [4] have recently shown that the Weinstock and Hersch–Payne–Schiffer inequalities are unstable, in the sense that there are domains very far from the disk — or from a union of k identical disks — with their kth normalised eigenvalue arbitrarily close to $2\pi k$. In fact, they prove the following result.

Theorem 1.4 ([4, Theorem 1.1])**.** *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^2$ be two bounded, conformally equivalent domains with smooth boundary. Then, there exists a sequence of open domains Ω^ε with uniformly bounded perimeter such that*

$$d_{Haus}(\partial\Omega^\varepsilon, \partial\Omega_1) \xrightarrow{\varepsilon\to 0} 0, \quad \textit{and}, \quad \textit{for all } k \in \mathbb{N},\ \Sigma_k(\Omega^\varepsilon) \xrightarrow{\varepsilon\to 0} \Sigma_k(\Omega_2).$$

Here, the domains Ω^ε are diffeomorphic to the original domains. They are obtained by a local perturbation of the boundary.

We investigate further flexibility results for the Steklov spectrum of domains in Euclidean space. Let $\Omega \subset \mathbb{R}^d$ be a connected bounded open set with Lipschitz boundary. For $\beta \in C(\overline{\Omega})$ positive, consider the weighted Neumann problem

$$\begin{cases} -\Delta f = \nu\beta f & \text{in } \Omega, \\ \partial_n f = 0 & \text{on } \partial\Omega. \end{cases}$$

The eigenvalues form a sequence $0 = \nu_0 < \nu_1(\Omega,\beta) \leqslant \nu_2(\Omega,\beta) \leqslant \ldots \nearrow \infty$. For $\beta \equiv 1$ we write $\nu_k(\Omega) := \nu_k(\Omega, 1)$ for the classical Neumann eigenvalue when there is no risk of confusion.

Theorem 1.5. *There is a family $\Omega^\varepsilon \subset \Omega$ of domains such that for each $k \in \mathbb{N}$,*

$$\sigma_k(\Omega^\varepsilon)\left|\partial\Omega^\varepsilon\right| \xrightarrow{\varepsilon\to 0} \nu_k(\Omega,\beta)\int_\Omega \beta\,\mathrm{d}\mathbf{x}, \qquad \nu_k(\Omega^\varepsilon,1) \xrightarrow{\varepsilon\to 0} \nu_k(\Omega,1), \qquad \left|\Omega^\varepsilon\right| \xrightarrow{\varepsilon\to 0} |\Omega|.$$

In many ways, the Neumann and Steklov problems have similar features. This has led to an investigation of bounds for one eigenvalue problem in terms of the other, see e.g. [12, 19]. It was previously thought that some universal inequalities between perimeter-normalised Steklov eigenvalues and area-normalised Neumann eigenvalues could exist. It is known from [26, Section 2.2] that normalised Steklov eigenvalues can be arbitrarily small while keeping the normalised Neumann eigenvalues bounded away from zero. We use Theorem 1.5 to prove that there are also domains with arbitrarily small area-normalised Neumann eigenvalues $\nu_k(\Omega)|\Omega|$, for which the Steklov eigenvalues are bounded away from zero.

Corollary 1.6. *There exists a sequence of planar domains Ω^ε such that the normalised Steklov eigenvalue $\Sigma_1(\Omega^\varepsilon) \xrightarrow{\varepsilon\to 0} 8\pi$ while for each $k \in \mathbb{N}$, the normalised Neumann eigenvalues satisfy $\nu_k(\Omega^\varepsilon)|\Omega^\varepsilon| \xrightarrow{\varepsilon\to 0} 0$.*

1.3. **Plan of the paper.** In Section 2 a general framework for the convergence of variational eigenvalues associated to a Radon measure is presented. This framework allows one to compare hard to relate eigenvalue problems on a given manifold.

In Section 3, we construct domains Ω^ε by removing small balls that are asymptotically uniformly densely distributed, with radii distributed according to the density β. This is in the spirit of deterministic homogenisation theory and follows the strategy of [8] for $\beta \equiv 1$. We prove Theorem 3.1, which is Theorem 1.5 for that specific family Ω^ε. This is done by applying the results of Section 2 thrice along with the methods developed in [9] for homogenisation on manifolds, in order to deal with the fact that the density is not constant.

The proof of Theorem 1.1 is presented in Section 4. Starting with a metric on the sphere for which the kth Laplace eigenvalue is almost maximal and removing a small disk will lead to an appropriate density on any simply connected domain, for which the kth inhomogeneous Neumann eigenvalue is almost $8k\pi$. We then use Theorem 1.5 to conclude.

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2. SPECTRAL CONVERGENCE

Eigenvalue convergence results are ubiquitous in the literature and the proofs of a large number of them follow similar steps. In the present section we formulate these steps explicitly in sufficient generality to allow direct application to many natural eigenvalue problems, including both Steklov and Laplace eigenvalues.

2.1. **Variational eigenvalues associated to a Radon measure.** We generalise to higher dimension the definition of eigenvalues associated to a measure, introduced in [18] for surfaces. Given a connected compact Riemannian manifold (M,g) with or without boundary and a Radon measure μ on M we define the variational eigenvalues $\lambda_k(M,g,\mu)$ in the following way. For any $u \in \mathrm{C}^\infty(M)$ such that u is not 0 in $\mathrm{L}^2(M,\mu)$, we define the Rayleigh quotient $R_g(u,\mu)$ by

$$R_g(u,\mu) := \frac{\displaystyle\int_M |\nabla u|_g^2 \,\mathrm{d}v_g}{\displaystyle\int_M u^2 \,\mathrm{d}\mu}.$$

The eigenvalues $\lambda_k(M,g,\mu)$ are then given by

$$\lambda_k(M,g,\mu) := \inf_{F_{k+1}} \sup_{u\in F_{k+1}\setminus\{0\}} R_g(u,\mu), \tag{3}$$

where the infimum is taken over all $(k+1)$-dimensional subspaces $F_{k+1} \subset \mathrm{C}^\infty(M)$ that remain $(k+1)$-dimensional in $L^2(M,\mu)$.

Example 2.1. Variational eigenvalues are very general. By choosing appropriate Radon measure, one recovers many well known spectral problems.

- If M is closed and $\mu = \mathrm{d}v_g$, the volume measure associated to g, then $\lambda_k(M,g,\mathrm{d}v_g)$ are classical Laplacian eigenvalues;
- More generally, if $\beta \in \mathrm{C}^\infty(M)$ is a positive function and $\mu = \beta\,\mathrm{d}v_g$, then $\lambda_k(M,g,\beta\,\mathrm{d}v_g)$ are eigenvalues of the a weighted Laplacian;
- If M is a compact manifold with boundary, then $\lambda_k(M,g,\mathrm{d}v_g)$ are Neumann eigenvalues;
- If M is a compact manifold with boundary and $\mu = \iota_*\mathrm{d}A_g$, the pushforward by inclusion of the induced volume measure on ∂M, then $\lambda_k(M,g,\mu)$ are Steklov eigenvalues;
- If $\mu = \iota_*\mathrm{d}A_g + \beta\,\mathrm{d}v_g$, then $\lambda_k(M,g,\mu)$ are eigenvalues associated with a dynamical boundary value problem, see [3, 8].

Remark 2.2. In [2], Anné–Post describe changes in the spectrum of the Laplacian under wild perturbations of manifolds, using *generalised resolvent convergence*, see [22], also sometimes called *E-convergence* [6]. In that setting, one compares for $j \in \{1,2\}$ the spectrum of operators A_j defined on "energy form domains" $\mathscr{H}_j^1$ dense in Hilbert spaces $\mathscr{H}_j$. This is done through intertwining operators $J_{jk} : \mathscr{H}_j \to \mathscr{H}_k$ and $J_{jk}^1 : \mathscr{H}_j^1 \to \mathscr{H}_k^1$. In the setting we describe, we may only have access to natural operators of the type J^1 between energy form domains but not J between the Hilbert spaces themselves. This means that in a sense, we study "very wild" perturbations of the domain of the Laplacian.

In [18] Kokarev studied properties of eigenvalues $\lambda_k(M,g,\mu)$. Below we describe some fundamental results from that paper and adapt the proofs to our setting. Let $\mathscr{L}(M,\mu)$ be the completion of $\mathrm{C}^\infty(M)$

with respect to the norm

$$\|u\|^2_{\mathscr{L}(M,\mu)} = \int_M u^2\,\mathrm{d}\mu + \int_M |\nabla u|_g^2\,\mathrm{d}v_g = \|u\|^2_{\mathrm{L}^2(M,\mu)} + \|\nabla u\|^2_{\mathrm{L}^2(M,g)}.$$

There is a natural map $T'_\mu : \mathscr{L}(M,\mu) \to \mathrm{L}^2(M,\mu)$ which comes from this completion. In the classical setting where μ is the Lebesgue measure on M, the map T'_μ is the the embedding of the Sobolev space $H^1(M) \subset L^2(M)$, while for $\mu = \iota_* \mathrm{d}A_g$, the map T'_μ is the trace operator $H^1(M) \to L^2(\partial M)$. For general measures, the space $\mathscr{L}(M,\mu)$ could be very different from the Sobolev space $\mathrm{H}^1(M)$ and the eigenvalues $\lambda_j(M,g,\mu)$ could lack natural properties one expects from eigenvalues. For that reason we restricts ourselves to a particular class of admissible measures.

Definition 2.3. A Radon measure is called admissible if the identity map on $\mathrm{C}^\infty(M)$ extends to a compact operator $T_\mu\colon \mathrm{H}^1(M) \to \mathrm{L}^2(M,\mu)$.

Proposition 2.4. *Suppose* μ *is not concentrated at a single point and* $\lambda_1(M,g,\mu) > 0$. *There is* $C_\mu > 0$ *such that for any* $u \in \mathrm{C}^\infty(M)$ *one has*

$$C_\mu^{-1}\,\|u\|^2_{\mathrm{H}^1(M,g)} \leqslant \|u\|^2_{\mathscr{L}(M,\mu)} \leqslant C_\mu\,\|u\|^2_{\mathrm{H}^1(M,g)}.$$

In particular, the identity map on $\mathrm{C}^\infty(M)$ *extends to an isomorphism between* $\mathrm{H}^1(M,g)$ *and* $\mathscr{L}(M,\mu)$.

Proof. Kokarev proved in [18, Lemma 2.5] that conditions of the proposition implies there is a well-defined bounded map $\mathrm{H}^1(M,g) \to \mathrm{L}^2(M,\mu)$. In particular,

$$\|u\|^2_{\mathrm{H}^1(M,g)} \geqslant C_\mu\,\|u\|^2_{\mathscr{L}(M,\mu)}.$$

To prove the opposite inequality we proceed as follows. First, recall the following Poincaré-type inequality found e.g. in [1, Lemma 8.3.1]. For any Riemannian manifold (M,g), there exists $C > 0$ such that for all $L \in \mathrm{H}^{-1}(M)$ with $L(1) = 1$ one has

$$\|u - L(u)\|_{\mathrm{L}^2(M)} \leqslant C\,\|L\|_{\mathrm{H}^{-1}(M)} \left(\int_M |\nabla u|_g^2\, d v_g\right)^{1/2} \tag{4}$$

for all $u \in \mathrm{H}^1(M)$.

Applying the adjoint of the embedding $\mathrm{H}^1(M,g) \to \mathrm{L}^2(\mu)$ to the constant function implies that $L(u) = \frac{1}{\mu(M)}\int u\,\mathrm{d}\mu$ is an element of $\mathrm{H}^{-1}(M,g)$. Thus, (4) implies

$$\begin{aligned}\int u^2\,\mathrm{d}v_g &\leqslant 2\int\left(u - \frac{1}{\mu(M)}\int u\,\mathrm{d}\mu\right)^2 \mathrm{d}v_g + 2\mathrm{Vol}(M,g)\left(\frac{1}{\mu(M)^2}\int u\,\mathrm{d}\mu\right)^2 \\ &\leqslant C_\mu \int |\nabla u|_g^2\,\mathrm{d}v_g + C'_\mu \int u^2\,\mathrm{d}\mu.\end{aligned} \tag{5}$$

□

Proposition 2.5. *The Radon measure* μ *not concentrated at a single point is admissible if and only if* T'_μ *is compact and* $\lambda_1(M,\mu,c) > 0$.

Proof. ($\Longleftarrow$) Follows from Proposition 2.4.

($\Longrightarrow$) Admissibility implies that the map $T_\mu\colon \mathrm{H}^1 \to \mathrm{L}^2(M,\mu)$ is well-defined. In particular, the proof of (5) follows through, i.e. the identity map on $C^\infty(M)$ extends to a map $j\colon \mathscr{L}(M,\mu) \to \mathrm{H}^1(M,g)$. Thus, $T'_\mu = T_\mu \circ j$ is compact as it is the composition of a bounded operator and a compact operator. We now prove that $\lambda_1(M,g,\mu) > 0$. Assume otherwise that $\lambda_1(M,g,\mu) = 0$. Then there exists a sequence of smooth functions u_n such that

$$\int_M u_n^2 \,\mathrm{d}\mu = 1, \quad \int_M |\nabla u_n|_g^2 \,\mathrm{d}v_g \to 0, \quad \int_M u_n \,\mathrm{d}\mu = 0.$$

By (5) the functions u_n are uniformly bounded in $\mathrm{H}^1(M)$. Since T_μ is compact, we can choose a subsequence such that u_n converge weakly in $\mathrm{H}^1(M)$ to $u \in \mathrm{H}^1(M)$ and strongly to $T_\mu(u)$ in $L^2(M,\mu)$. Then u satisfies $\|\nabla u\|_{\mathrm{L}^2(M)} = 0$, therefore, u is constant function, thus, $T_\mu(u)$ is a constant, which contradicts the fact that

$$\int_M T_\mu(u)^2 \,\mathrm{d}\mu = 1, \quad \int_M T_\mu(u) \,\mathrm{d}\mu = 0.$$

$\square$

We see that for admissible measures $\mathscr{L}(M,\mu)$ is simply $\mathrm{H}^1(M)$ with a different norm, albeit a more convenient one for our purposes. The following proposition states that the eigenvalues of admissible measures possess all the natural properties one expects from eigenvalues of an operator of Laplace-type. We include the following two statements and their proof for completeness.

Proposition 2.6. *Let μ be an admissible measure. Then one has*

$$0 = \lambda_0(M,g,\mu) < \lambda_1(M,g,\mu) \leqslant \lambda_2(M,g,\mu) \leqslant \ldots \nearrow \infty;$$

i.e. the first eigenvalue is positive, the multiplicity of each eigenvalue is finite, and the eigenvalues tend to $+\infty$. Moreover, there exists an orthogonal basis of eigenfunctions $f_j \in \mathscr{L}(M,\mu)$ satisfying

$$\int_M \nabla f_j \cdot \nabla u \,\mathrm{d}v_g = \lambda_j(M,g,\mu) \int_M f_j u \,\mathrm{d}\mu \tag{6}$$

for all $u \in \mathscr{L}(M,\mu)$.

Proof. The proof is standard and is similar to [18, Proposition 1.3]. We first prove by induction on k that there exists eigenfunction corresponding to $\lambda_k(M,g,\mu)$. For $k = 0$ one has $\lambda_0(M,g,\mu) = 0$ and $f_0 = 1$, which obviously satisfies (6). Suppose that we have already found the eigenfunctions $f_0,\ldots,f_k$, let V_{k+1} be their span. By definition, there exists a sequence of $(k+2)$-dimensional spaces $F^n \subset C^\infty(M)$ such that

$$\sup_{u \in F^n \setminus \{0\}} R_g(u,\mu) \to \lambda_{k+1}(M,g,\mu).$$

Let $u_n \in F^n \setminus \{0\}$ be $\mathrm{d}\mu$-orthogonal to V_{k+1}. After possible rescaling of u_n one has

$$\int_M u_n^2 \,\mathrm{d}\mu = 1, \qquad \int_M |\nabla u_n|_g^2 \,\mathrm{d}v_g \to \lambda_{k+1}(M,g,\mu).$$

Thus, $\{u_n\}$ is a bounded sequence in $\mathscr{L}(M,\mu)$, hence there exists $f_{k+1} \in \mathscr{L}(M,\mu)$ such that, up to a choice of subsequence, $u_n \rightharpoonup f_{k+1}$ weakly in $\mathscr{L}(M,\mu)$. Since T'_μ is compact the convergence is strong in $\mathrm{L}^2(M,\mu)$. Therefore, f_{k+1} satisfies

$$\int_M f_{k+1}^2 \,\mathrm{d}\mu = 1, \quad \int_M |\nabla f_{k+1}|_g^2 \,\mathrm{d}v_g \leqslant \lambda_{k+1}(M,g,\mu), \quad \int_M f_{k+1} f \,\mathrm{d}\mu = 0$$

for all $f \in V_{k+1}$. Assume that $R_g(f_{k+1},\mu) < \lambda_{k+1}(M,g,\mu)$. Let $j \leq k$ be such that $\lambda_j(M,g,\mu) < \lambda_{j+1}(M,g,\mu) = \lambda_{k+1}(M,g,\mu)$. Consider $V = V_{j+1} \oplus \mathbb{R} f_{k+1}$, then for any $v = v_1 + v_2 \in V$ one has

$$\begin{aligned}\int |\nabla v|_g^2 \, \mathrm{d}v_g &= \int |\nabla v_1|_g^2 + |\nabla v_2|_g^2 \, \mathrm{d}v_g \\ &< \lambda_{j+1}(M,g,\mu) \int v_1^2 + v_2^2 \, \mathrm{d}\mu \\ &= \lambda_{j+1}(M,g,\mu) \int v^2 \, \mathrm{d}\mu,\end{aligned}$$

where in the first equality we used (6) for functions in V_{j+1}. We obtain a contradiction, since

$$\lambda_{j+1}(M,g,\mu) \leq \sup_{v \in V \setminus \{0\}} R_g(v,\mu) < \lambda_{j+1}(M,g,\mu).$$

Thus, we proved $R_g(f_{k+1},\mu) = \lambda_{k+1}(M,g,\mu)$. Finally, we show that f_{k+1} is the corresponding eigenfunction, i.e. that it satisfies (6). Indeed, if $u \in V_{k+1}$ one has that $t \mapsto R_g(f_{k+1} + tu, \mu)$ has a maximum point at $t = 0$. If $u \in V_{k+1}^{\perp}$, then $t \mapsto R_g(f_{k+1} + tu, \mu)$ has a minimum at $t = 0$. Regardless, $t \mapsto R_g(f_{k+1} + tu, \mu)$ has a critical point at $t = 0$. Differentiating this at $t = 0$ yields (6).

A similar compactness argument shows that given $\lambda > 0$ any subspace $V \subset \mathscr{L}(M,\mu)$ satisfying

$$\sup_{u \in V \setminus \{0\}} R_g(u,\mu) \leq \lambda$$

has to be finite dimensional in $\mathrm{L}^2(M,\mu)$. As a result, the number of eigenvalues $\lambda_j(M,g,\mu)$ below λ is finite, which, in particular, implies that the multiplicity of each eigenvalue is finite and, moreover, that $\{f_j\}$ form a basis. □

2.2. **Continuity of eigenvalues.** The eigenvalues $\lambda_k(M,g,\mu)$ are not necessarily continuous under weak-$*$ convergence but they are always upper-semicontinuous, see [18, Proposition 1.1]. We also include the proof in this context for completeness, but it is the same in essence.

Proposition 2.7. *Let (M,g) be a Riemannian manifold and assume $\mu_n \xrightharpoonup{*} \mu$. Then*

$$\limsup_{n \to \infty} \lambda_k(M,g,\mu_n) \leq \lambda_k(M,g,\mu)$$

Proof. Let $\varepsilon > 0$ be arbitrary. Let $F \subset C^\infty(M)$ be a $(k+1)$-dimensional subspace that remains $(k+1)$-dimensional in $\mathrm{L}^2(M,\mu)$ and

$$\sup_{u \in F \setminus \{0\}} R_g(u,\mu) \leq \lambda_k(M,g,\mu) + \varepsilon.$$

Convergence $\mu_n \xrightharpoonup{*} \mu$ implies that for large n the subspace F is $(k+1)$-dimensional in $L^2(M,\mu_n)$ and

$$\lim_{n \to \infty} \sup_{u \in F \setminus \{0\}} R_g(u,\mu_n) = \sup_{u \in F \setminus \{0\}} R_g(u,\mu).$$

As a result, for large n one has

$$\lambda_k(M,g,\mu_n) \leq \sup_{u \in F \setminus \{0\}} R_g(u,\mu_n) \leq \lambda_k(M,g,\mu) + 2\varepsilon$$

□

For many applications it is important to establish continuity of eigenvalues. To the best of the authors' knowledge there is no sufficiently general condition that guarantees continuity of $\lambda_k(M,g,\mu)$. As an example, we note that all examples of convergence covered in the present paper fail the integral distance convergence criterion given in [18, Section 4.2]. Nevertheless, many proofs in very different settings follow the same structure which we outline below.

Let μ_n be a collection of Radon measures on (M,g) such that

(**M1**) $\mu_n \overset{*}{\rightharpoonup} \mu$;
(**M2**) the measures μ, μ_n are admissible for all n.

The condition (**M2**) guarantees the existence of the μ_n-orthonormal collection of eigenfunctions f_j^n associated with $\lambda_j(M,g,\mu_n)$.

We now need the following three conditions on the eigenfunctions.

(**EF1**) The functions f_j^n are bounded in $\mathrm{L}^2(M,\mu)$.
(**EF2**) For all $v \in \mathscr{L}(M,\mu)$, the functions f_j^n satisfy

$$\lim_{n\to\infty}\left|\langle f_j^n, v\rangle_{\mathrm{L}^2(M,\mu)} - \langle f_j^n, v\rangle_{\mathrm{L}^2(M,\mu_n)}\right| = 0.$$

(**EF3**) For every $j,k\in\mathbb{N}$, the functions f_j^n, f_k^n satisfy

$$\lim_{n\to\infty}\left|\langle f_j^n, f_k^n\rangle_{\mathrm{L}^2(M,\mu)} - \langle f_j^n, f_k^n\rangle_{\mathrm{L}^2(M,\mu_n)}\right| = 0.$$

Condition (**EF1**) is there to ensure that $\left\{f_j^n\right\}$ is bounded in $\mathscr{L}(M,\mu)$. One then has that, up to a subsequence, $f_j^n \rightharpoonup f_j$ weakly in $\mathscr{L}(M,\mu)$ and $\lambda_j(M,g,\mu_n)\to\lambda_j$.

Proving that Conditions (**EF2**) and (**EF3**) hold is usually the hardest part. The former implies that f_j are eigenfunctions associated with (M,g,μ) with the corresponding eigenvalues λ_j. At this point it is unclear whether λ_j is indeed the j-th eigenvalue $\lambda_j(M,g,\mu)$. In order to establish this one usually proves the last condition (**EF3**) and provides an argument showing that (**EF3**) implies $\lambda_j(M,g,\mu)=\lambda_j$. This last argument is the same regardless of the eigenvalues problem at hand. We formalize this procedure in the following proposition.

Proposition 2.8. *Assume that the Radon measures μ_n, μ on (M,g) satisfy conditions (**M1**)–(**M2**), and that the eigenfunctions associated with μ_n satisfy conditions (**EF1**)–(**EF3**). Then*

$$\lim_{n\to\infty}\lambda_j(M,g,\mu_n) = \lambda_j(M,g,\mu),$$

and, up to a choice of subsequence,

$$\lim_{n\to\infty} f_j^n = f_j,$$

strongly in $\mathscr{L}(M,\mu)$. If $\lambda_j(M,g,\mu)$ is simple, the convergence is along the whole sequence.

Proof. For each fixed j

$$\left\|f_j^n\right\|^2_{\mathscr{L}(M,\mu)} = \left\|f_j^n\right\|^2_{\mathrm{L}^2(M,\mu)} + \lambda_j(M,g,\mu_n). \tag{7}$$

By Proposition 2.7 along with Condition (**M1**), up to a subsequence $\lambda_j(M,g,\mu_n)\to\lambda_j\leqslant\lambda_j(M,g,\mu)$. Combining (7) with (**EF1**) implies that up to a subsequence, $f_j^n\rightharpoonup f_j$ weakly in $\mathscr{L}(M,\mu)$.

Condition (**EF2**) implies that f_j is an eigenfunction associated with (M,g,μ) and corresponding eigenvalue λ_j. Indeed, by weak convergence we have that for any $v \in \mathscr{L}(M,\mu)$,

$$\int_M \nabla f_j^n \cdot \nabla v \, \mathrm{d}v_g \xrightarrow{n\to\infty} \int_M \nabla f_j \cdot \nabla v \, \mathrm{d}v_g.$$

On the other hand we have that

$$\left|\langle f_j, v\rangle_{\mathrm{L}^2(M,\mu)} - \langle f_j^n, v\rangle_{\mathrm{L}^2(M,\mu_n)}\right| \leqslant \left|\langle f_j - f_j^n, v\rangle_{\mathrm{L}^2(M,\mu)}\right| + \left|\langle f_j^n, v\rangle_{\mathrm{L}^2(M,\mu)} - \langle f_j^n, v\rangle_{\mathrm{L}^2(M,\mu_n)}\right|.$$

By Condition (**M2**), f_j^n converges strongly in $\mathrm{L}^2(M,\mu)$ so that the first term on the righthand side converges to 0 while Condition (**EF2**) implies that the second term converges to 0.

We can now prove that the limit sequence f_j is orthonormal. Indeed,

$$\langle f_j, f_k\rangle_{\mathrm{L}^2(M,\mu)} = \langle f_j^n, f_k^n\rangle_{\mathrm{L}^2(M,\mu)} + \langle f_j, f_k - f_k^n\rangle_{\mathrm{L}^2(M,\mu)} + \langle f_j - f_j^n, f_k^n\rangle_{\mathrm{L}^2(M,\mu)}.$$

Strong convergence in $\mathrm{L}^2(M,\mu)$, Condition (**EF1**) and the Cauchy-Schwarz inequality imply that the last two terms on the righthand side converge to 0, whereas Condition (**EF3**) implies that the first term converges to δ_{jk}.

To prove that $\lambda_j(M,g,\mu) \leqslant \lambda_j$, we use the space $F_{j+1} = \mathrm{span}\{f_0,\dots,f_j\}$ as a test-space in (3). For any $f = \sum a_i f_i \in F_j$ one has

$$\frac{\displaystyle\int_M |\nabla f|_g^2 \, \mathrm{d}v_g}{\displaystyle\int_M f^2 \, \mathrm{d}\mu} = \frac{\sum_{i=0}^j \lambda_i a_i^2}{\sum_{i=0}^j a_i^2} \leqslant \lambda_j \frac{\sum_{i=0}^j a_i^2}{\sum_{i=0}^j a_i^2} = \lambda_j,$$

where Condition (**EF3**) is used in the first equality. Finally, note that weak convergence and convergence of the norms implies strong convergence, and it follows from (7) that we do indeed have convergence of the norms. $\square$

It is often the case that one would like to study the stability properties of eigenvalue problems defined on varying domains. Below we provide appropriate modifications to the aforementioned setup and conditions (**M1**), (**M2**) and (**EF1**)– (**EF3**).

Let μ_n be a collection of Radon measures on (M,g) and $\Omega_n \subset M$ be a collection of domains in M viewed as Riemannian manifolds with the metric induced from M. We use the same notations g, μ_n for their restrictions to Ω_n. Suppose that

(**M1***) $\mathrm{supp}(\mu_n) \subset \overline{\Omega_n}$ and $\mu_n \xrightarrow{*} \mu$ as measures on M;
(**M2***) the measures μ_n, μ are admissible on Ω_n, M respectively.
(**M3***) there is a bounded extension map $J_n : \mathscr{L}(\Omega_n,\mu_n) \to \mathscr{L}(M,\mu_n)$.

The condition (**M2***) guarantees for every n the existence of the μ_n-orthonormal collection of eigenfunctions $f_j^n \in \mathscr{L}(\Omega_n,\mu_n)$ associated with $\lambda_j(\Omega_n,g,\mu_n)$. The map J_n is often built using harmonic functions. Note that the extensions $J_n f_j^n$ remain μ_n orthonormal. We then have the following three conditions on the extended eigenfunctions.

(**EF1***) The functions $J_n f_j^n$ are bounded in $\mathrm{L}^2(M,\mu)$.
(**EF2***) For all $v \in \mathscr{L}(M,\mu)$, the functions $J_n f_j^n$ satisfy

$$\lim_{n\to\infty} \left|\langle J_n f_j^n, v\rangle_{\mathscr{L}(M,\mu)} - \langle f_j^n, v\rangle_{\mathscr{L}(\Omega,\mu_n)}\right| = 0$$

(**EF3***) For every $j,k \in \mathbb{N}$, the functions $J_n f_j^n$, $J_n f_k^n$ satisfy

$$\lim_{n\to\infty} \left| \langle J_n f_j^n, J_n f_k^n \rangle_{\mathrm{L}^2(M,\mu)} - \langle f_j^n, f_k^n \rangle_{\mathrm{L}^2(\Omega_n,\mu_n)} \right| = 0.$$

Note that in condition (**EF2***), we now have to assume convergence of the inner products on $\mathscr{L}$ rather than simply on L^2.

Proposition 2.9. *Assume that the Radon measures μ_n, μ on (M,g) and domains $\Omega_n \subset M$ satisfy conditions (**M1***)– (**M3***) and that the eigenfunctions associated with μ_n satisfy (**EF1***)– (**EF3***). Then*

$$\lim_{n\to\infty} \lambda_j(\Omega_n, g, \mu_n) = \lambda_j(M, g, \mu),$$

and, up to a choice of subsequence,

$$\lim_{n\to\infty} J_n f_j^n = f_j,$$

weakly in $\mathscr{L}(M,\mu)$. If $\lambda_j(M,g,\mu)$ is simple, the convergence is along the whole sequence. Finally, if

$$\limsup_{n\to\infty} \left| \left\| J_n f_j^n \right\|_{\mathscr{L}(M,\mu_n)} - \left\| f_j^n \right\|_{\mathscr{L}(\Omega_n,\mu_n)} \right| = 0, \tag{8}$$

the convergence is strong in $\mathscr{L}(M,\mu)$.

Proof. The inequality $\lambda_j \geqslant \lambda_j(M,g,\mu)$ and the convergence of the eigenfunctions is proved in the same way as Proposition 2.8. To prove the reverse inequality, note that by definition (3) one has $\lambda_j(\Omega_n, g, \mu_n) \leqslant \lambda_j(M, g, \mu_n)$. Taking the limit $n \to \infty$ and using Proposition 2.7 one has

$$\lambda_j \leqslant \limsup_{n\to\infty} \lambda_j(M, g, \mu_n) \leqslant \lambda_j(M, g, \mu).$$

□

3. HOMOGENISATION

In this section, we fix a domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary, β a non-negative, Riemann integrable function, $\mu_\beta = \beta\, \mathrm{d}\mathbf{x}$ and g_0 the flat metric. We prove the following theorem.

Theorem 3.1. *There is a family $\Omega^\varepsilon \subset \Omega$ such that for $\mu^\varepsilon := \iota_* \mathrm{d}A^\varepsilon$, $\widetilde{\mu}^\varepsilon := \mathrm{d}\mathbf{x}|_{\Omega^\varepsilon}$ and all $j \in \mathbb{N}$,*

$$\sigma_j(\Omega^\varepsilon) \left|\Omega^\varepsilon\right| = \lambda_j(\Omega^\varepsilon, g_0, \mu^\varepsilon)\mu^\varepsilon(\Omega^\varepsilon) \xrightarrow{\varepsilon\to 0} \lambda_j(\Omega, g_0, \mu)\mu_\beta(\Omega)$$

and

$$\nu_j(\Omega^\varepsilon) \left|\Omega^\varepsilon\right| = \lambda_j(\Omega^\varepsilon, g_0, \widetilde{\mu}^\varepsilon)\widetilde{\mu}^\varepsilon(\Omega^\varepsilon) \xrightarrow{\varepsilon\to 0} \lambda_j(\Omega^\varepsilon, g_0, \mathrm{d}\mathbf{x}) \left|\Omega\right|. \tag{9}$$

The proof is split into three parts. In the first part, we prove that as $\alpha \to 0$, the measures $\mu_\beta^\alpha := \alpha \iota_* \,\mathrm{d}A + \beta\, \mathrm{d}\mathbf{x} \xrightharpoonup{*} \mu_\beta$ and respects conditions (**M1**)–(**M2**) and (**EF1**)–(**EF3**). We make the observation that

$$\lambda_j(\Omega, g_0, \mu_\beta^\alpha) = \frac{1}{\alpha} \lambda_j(\Omega, g_0, \alpha^{-1}\mu_\beta^\alpha).$$

The second step is to construct the domains Ω^ε such that $\mu^\varepsilon \xrightharpoonup{*} \alpha^{-1}\mu_\beta^\alpha$ and $\widetilde{\mu}^\varepsilon \xrightharpoonup{*} \mathrm{d}\mathbf{x}$. It will be clear from the construction that conditions (**M1***)–(**M3***) are satisfied. Finally, in the last part we show that their eigenfunctions satisfy conditions (**EF1***)–(**EF3***), Theorem 3.1 is then a specialisation of Proposition 2.9, along with the fact that $\mu_\beta(\Omega) = \mu_\beta^\alpha(\Omega) + O(\alpha)$. We note that (9) could be deduced by an appropriate modification of the proofs in [23] or [2], however this would require introducing new concepts whereas the results from Section 2 can prove both at the same time. This also puts an emphasis on the fact that it is achieved for the same domains.

We start with the following proposition.

Proposition 3.2. *Let Ω be a Riemannian manifold with boundary, $\beta \in \mathrm{C}(\overline{\Omega})$, and $\alpha > 0$. Denote $\mu = \beta\,\mathrm{d}v_g$ and $\mu^\alpha = \alpha \mathrm{d}A_g + \mu$. Then,*

$$\lim_{\alpha\to 0} \lambda_j(\Omega, g, \mu^\alpha) = \lambda_j(\Omega, g, \mu).$$

Proof. It is sufficient to prove that Conditions (**M1**), (**M2**) and (**EF1**)–(**EF3**) are satisfied. It is easy to see that $\mu^\alpha \overset{*}{\rightharpoonup} \mu$, and that for all α, the maps T_{μ^α}, T_μ are compact, so that Conditions (**M1**) and (**M2**) are satisfied. Denote by $f_j^{(\alpha)}$ the eigenfunction associated with $\lambda_j(\Omega, g, \mu^\alpha)$. Condition (**EF1**) is satisfied since, by normalisation,

$$1 \geqslant \int_\Omega (f_j^\alpha)^2 \,\mathrm{d}\mu.$$

Condition (**EF2**) is satisfied since for all $v \in \mathscr{L}(\Omega, \mu)$

$$\left|\langle f_j^{(\alpha)}, v\rangle_{\mathrm{L}^2(\Omega,\mu^\alpha)} - \langle f_j^{(\alpha)}, v\rangle_{\mathrm{L}^2(\Omega,\mu)}\right| = \alpha \left|\int_{\partial\Omega} f_j^\alpha\, v \,\mathrm{d}A\right| \leqslant \alpha \,\|v\|_{\mathrm{L}^2(\partial\Omega, \mathrm{d}A)}\,.$$

The last term converges to 0 as $\alpha \to 0$ by boundedness of the Sobolev trace. Similarly, condition (**EF3**) holds since

$$\left|\langle f_j^{(\alpha)}, f_k^{(\alpha)}\rangle_{\mathrm{L}^2(\Omega,\mu^\alpha)} - \langle f_j^{(\alpha)}, f_k^{(\alpha)}\rangle_{\mathrm{L}^2(\Omega,\mu)}\right| = \left|\int_{\partial\Omega} f_j^{(\alpha)} f_k^{(\alpha)} \,\mathrm{d}A\right| \leqslant \alpha.$$

By Proposition 2.8 our claim holds. □

3.1. **Construction of perforated sets.** We construct the domains $\Omega^\varepsilon \subset \Omega$ in the spirit of deterministic homogenisation theory. For $\mathbf{k} \in \mathbb{Z}^d$, consider the cubes

$$Q_{\mathbf{k}}^\varepsilon := \varepsilon\mathbf{k} + \left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right]^d \subset \mathbb{R}^d$$

and define

$$\mathbf{I}^\varepsilon := \left\{\mathbf{k} \in \mathbb{Z}^d : Q_{\mathbf{k}}^\varepsilon \subset \Omega\right\}.$$

We set

$$r_{\mathbf{k}}^\varepsilon := \left(\frac{\beta(\varepsilon\mathbf{k})}{a_d}\varepsilon^d\right)^{\frac{1}{d-1}} \qquad \text{and} \qquad \mathbf{T}^\varepsilon := \bigcup_{\mathbf{k}\in I^\varepsilon} B\left(\varepsilon\mathbf{k}, r_{\mathbf{k}}^\varepsilon\right),$$

where a_d is the area of the unit sphere in $\mathbb{R}^d$ and where by convention, we put $B(\mathbf{x}, 0) = \varnothing$ for any $\mathbf{x}$. We set

$$\widetilde{\mathbf{I}}^\varepsilon := \left\{\mathbf{k} \in \mathbf{I}^\varepsilon : \beta(\mathbf{k}\varepsilon) \neq 0\right\}$$

and $\Omega^\varepsilon := \Omega \setminus \overline{\mathbf{T}^\varepsilon}$, and make the following observations.

- For all $\varepsilon, \mathbf{k}$,

$$\min_{\mathbf{x}\in\Omega}\left(\frac{\beta(\mathbf{x})}{a_d}\right)\varepsilon^d \leqslant (r_{\mathbf{k}}^\varepsilon)^{d-1} \leqslant \max_{\mathbf{x}\in\Omega}\left(\frac{\beta(\mathbf{x})}{a_d}\right)\varepsilon^d$$

- The number of holes satisfies $\#\widetilde{\mathbf{I}}^\varepsilon \ll \varepsilon^{-d}$ as $\varepsilon \to 0$.

– The total boundary area of the holes satisfies

$$\left|\partial \mathbf{T}^{\varepsilon}\right| = \sum_{\mathbf{k}\in\widetilde{\mathbf{I}}^{\varepsilon}} a_d (r_{\mathbf{k}}^{\varepsilon})^{d-1} \sim \int_{\Omega} \beta \,\mathrm{d}\mathbf{x},$$

and $\mu^{\varepsilon} \xrightharpoonup{*} \mu := \mu_{\beta} + \iota_* \,\mathrm{d}A$ so that they satisfy Condition (**M1***).

– The total volume of the holes satisfies

$$\left|\mathbf{T}^{\varepsilon}\right| = \sum_{\mathbf{k}\in\widetilde{\mathbf{I}}^{\varepsilon}} d a_d r_{\varepsilon}^{d} = O\left(r_{\varepsilon}\right) \tag{10}$$

and $\widetilde{\mu}^{\varepsilon} \xrightharpoonup{*} \mathrm{d}\mathbf{x}$, so that they also satisfy condition (**M1***).

– It is a standard fact that the trace maps $T'_{\mu^{\varepsilon}}$ and the Sobolev embeddings $T'_{\widetilde{\mu}^{\varepsilon}}$, T'_{μ} are compact. Furthermore, the first Steklov and the first Neumann eigenvalue on Lipschitz domains are always positive so that Condition (**M2***) is met in both cases.

– Denoting by $J^{\varepsilon} : \mathscr{L}(\Omega^{\varepsilon}, \mu^{\varepsilon}) \to \mathscr{L}(\Omega, \mu^{\varepsilon})$ the map extending harmonically in $\mathbf{T}^{\varepsilon}$, we have that J^{ε} is bounded independently of ε, see [23, Example 1, page 40]. Condition (**M3***) is therefore satisfied.

We may now turn to the associated eigenpairs.

3.2. **Convergence properties of the Steklov eigenpair.** We recall from Proposition 2.4 that the norm associated with $\mathscr{L}(\Omega,\mu)$ is equivalent to the usual $\mathrm{H}^1(\Omega)$ norm. For convenience, we state the results in terms of $\mathscr{L}(\Omega,\mu)$ terms, but the reader interested in the standard Euclidean setting can interpret them as $\mathrm{H}^1(\Omega)$ convergence results.

We denote by $\sigma_j^{(\varepsilon)} := \lambda_k(\Omega^{\varepsilon}, g_0, \mu^{\varepsilon})$ the Steklov eigenvalues of Ω^{ε} and by $u_j^{(\varepsilon)}$ the associated Steklov eigenfunctions. It follows from Proposition 2.7 that for each j, $\sigma_j^{(\varepsilon)}$ is bounded in ε. We also write $U_j^{(\varepsilon)} := J^{\varepsilon} u_j^{(\varepsilon)}$, which enjoys the following properties, proven in [8, Section 3].

– As $\varepsilon \to 0$,

$$\begin{aligned}\int_{\Omega} \left|\nabla U_j^{(\varepsilon)}\right|^2 \mathrm{d}\mathbf{x} &= \left(1 + O_{\beta}\left(\varepsilon^{\frac{1}{d-1}}\right)\right) \int_{\Omega^{\varepsilon}} \left|\nabla u_j^{(\varepsilon)}\right|^2 \mathrm{d}\mathbf{x} \\ &= \sigma_j^{(\varepsilon)} + O_{\beta,j}\left(\varepsilon^{\frac{1}{d-1}}\right).\end{aligned} \tag{11}$$

– There is a $C > 0$ depending only on β such that for all ε small enough,

$$\left\| U_j^{(\varepsilon)} \right\|_{\mathrm{L}^{\infty}(\Omega)} \leqslant C \tag{12}$$

Note that these properties were proved under the condition that β is a constant function. However, the proofs are local in nature and go through under the assumption that there is C such that $\beta(\mathbf{x}) < C$ for all $\mathbf{x} \in \Omega$, which is the case here by assumption.

Proposition 3.3. *As $\varepsilon \to 0$, $\sigma_j^{(\varepsilon)} \to \lambda_j(\Omega, g_0, \mu)$, where $\mu = \iota_* \,\mathrm{d}A + \mu_{\beta}^{\alpha}$. The harmonic extension $U_j^{(\varepsilon)}$ of the eigenfunctions converges strongly in $\mathscr{L}(\Omega,\mu)$ to the eigenfunctions associated with $\lambda_j(\Omega, g_0, \mu)$.*

Note that strong H^1 convergence is not a feature usually present in homogenisation theory and it depends on the specific eigenfunctions we are working with. In order to prove this proposition, we will prove in turn that Conditions (**EF1***)–(**EF3***) are satisfied.

Lemma 3.4. *The family $\left\{U_j^{(\varepsilon)}\right\}$ is bounded in $\mathrm{L}^2(\Omega,\mu)$. In other words, Condition (**EF1***) is satisfied.*

Proof. From (12),

$$\left\| U_j^{(\varepsilon)} \right\|_{\mathrm{L}^2(\Omega,\mu)} \leqslant \mu(\Omega)^{1/2} \left\| U_j^{(\varepsilon)} \right\|_{\mathrm{L}^\infty(\Omega)} = O_{\beta,\Omega,j}(1).$$

□

Proposition 3.5. *Let $j \in \mathbb{N}$ and $v \in \mathscr{L}(M,\mu)$. The limit*

$$\lim_{\varepsilon\to 0} \left| \langle U_j^{(\varepsilon)}, v\rangle_{\mathscr{L}(M,\mu)} - \langle u_j^{(\varepsilon)}, v\rangle_{\mathscr{L}(\Omega,\mu^\varepsilon)} \right| = 0$$

*holds. In other words, Condition (**EF2***) is satisfied.*

Proof. Observe first that the family of functionals $v \mapsto \langle U_j^{(\varepsilon)}, v\rangle_{\mathscr{L}(M,\mu)} \in \mathscr{L}(M,\mu)^*$ is bounded uniformly in ε, this follows from the $\mathrm{L}^\infty(M)$ bound on $U_j^{(\varepsilon)}$ in (12) and boundedness of the Dirichlet energy from Condition (**M1***). We therefore suppose that v belongs to the dense subspace $\mathrm{C}^\infty(\Omega)$.

It follows from weak $\mathscr{L}(\Omega,\mu)$ convergence and the Hölder inequality

$$\begin{aligned} \int_\Omega \nabla U_j^{(\varepsilon)} \cdot \nabla v \,\mathrm{d}\mathbf{x} - \int_{\Omega^\varepsilon} \nabla u_j^{(\varepsilon)} \cdot \nabla v \,\mathrm{d}\mathbf{x} &= \int_{\mathbf{T}^\varepsilon} \nabla U_j^{(\varepsilon)} \cdot \nabla v \,\mathrm{d}\mathbf{x} + o_{j,\mu,v}(1) \\ &= o_{j,\mu,v}(1). \end{aligned} \tag{13}$$

It remains to show that

$$\langle U_j^{(\varepsilon)}, v\rangle_{\mathscr{L}(\Omega,\mu)} - \langle u_j^{(\varepsilon)}, v\rangle_{\mathscr{L}(\Omega^\varepsilon,\mu^\varepsilon)} = \int_\Omega U_j^{(\varepsilon)} v \beta \,\mathrm{d}\mathbf{x} - \int_{\partial\mathbf{T}^\varepsilon} u_j^{(\varepsilon)} v \,\mathrm{d}A$$

converges to 0 as $\varepsilon \to 0$. To obtain convergence of this last term, we cannot use standard techniques from homogenisation theory where integrals are studied through an auxiliary function defined in a reference cell, this is due to β not being constant. We emulate the construction found in the proof of [9, Proposition 5.2]. For $\mathbf{k} \in \widetilde{\mathbf{I}}^\varepsilon$, define $\psi_{\mathbf{k}}^\varepsilon \in \mathrm{H}^1(Q_{\mathbf{k}}^\varepsilon)$ as the solution to the weak variational problem

$$\forall v \in \mathrm{H}^1(Q_{\mathbf{k}}^\varepsilon \setminus B(\varepsilon\mathbf{k}, r_{\mathbf{k}}^\varepsilon)), \qquad \int_{Q_{\mathbf{k}}^\varepsilon} \nabla\psi_{\mathbf{k}}^\varepsilon \cdot \nabla v \,\mathrm{d}\mathbf{x} = -c_{\varepsilon,\mathbf{k}} \int_{Q_{\mathbf{k}}^\varepsilon} v \,\mathrm{d}\mathbf{x} + \int_{\partial B_{r_\varepsilon}(\varepsilon\mathbf{k})} v \,\mathrm{d}A.$$

For $\psi_{\mathbf{k}}^\varepsilon$ to exist, it is necessary and sufficient (see [24, Theorem 5.7.7]) that

$$c_{\varepsilon,\mathbf{k}} = \frac{\beta(\varepsilon\mathbf{k})\varepsilon^2}{\varepsilon^2 + O_\beta\left(\varepsilon^4\right)} = \beta(\varepsilon\mathbf{k}) + O_\beta\left(\varepsilon^2\right).$$

The solution is unique if we require that $\int_{Q_{\mathbf{k}}^\varepsilon} \psi_{\mathbf{k}}^\varepsilon \,\mathrm{d}\mathbf{x} = 0$. It satisfies (see [9, Equation 25])

$$\left\| \nabla\psi_{\mathbf{k}}^\varepsilon \right\|_{\mathrm{L}^2(Q_{\mathbf{k}}^\varepsilon \setminus B(\varepsilon\mathbf{k}, r_{\mathbf{k}}^\varepsilon))} \ll_\beta \varepsilon^{d/2}.$$

For all $v \in \mathrm{C}^\infty(\Omega)$ we have that

$$\int_{\partial\mathbf{T}^\varepsilon} u_j^{(\varepsilon)} v \,\mathrm{d}A = \sum_{\mathbf{k}\in\widetilde{\mathbf{I}}^\varepsilon} \int_{Q_{\mathbf{k}}^\varepsilon \setminus B(\varepsilon\mathbf{k}, r_\varepsilon)} \nabla\psi_{\mathbf{k}}^\varepsilon \cdot \nabla(u_j^{(\varepsilon)} v) + c_{\varepsilon,\mathbf{k}} u_j^{(\varepsilon)} v \,\mathrm{d}\mathbf{x}.$$

The first term on the righthand side converges to 0. Indeed, it the generalised Hölder inequality yields

$$\begin{aligned} \sum_{\mathbf{k}\in\widetilde{\mathbf{I}}^\varepsilon} \int_{Q_{\mathbf{k}}^\varepsilon \setminus B(\varepsilon\mathbf{k}, r_\varepsilon)} \nabla\psi_{\mathbf{k}}^\varepsilon \nabla(u_j^{(\varepsilon)} v) \,\mathrm{d}\mathbf{x} &\leqslant \sum_{\mathbf{k}\in\widetilde{\mathbf{I}}^\varepsilon} \left\| \nabla\psi_{\mathbf{k}}^\varepsilon \right\|_{\mathrm{L}^2(Q_{\mathbf{k}}^\varepsilon \setminus B(\varepsilon\mathbf{k}, r_\varepsilon))} \| v \|_{\mathrm{C}^1(\Omega)} \left\| U_j^{(\varepsilon)} \right\|_{\mathrm{H}^1(Q_{\mathbf{k}}^\varepsilon)} \\ &\ll_{\beta,v} \varepsilon^{d/2} \left\| U_j^{(\varepsilon)} \right\|_{\mathrm{H}^1(\Omega)}. \end{aligned} \tag{14}$$

Strong $\mathrm{L}^2(\Omega,\mu)$ convergence on the second term implies that

$$\int_\Omega U_j^{(\varepsilon)} v\beta\,\mathrm{d}\mathbf{x} - \sum_{\mathbf{k}\in\tilde{\mathbf{I}}^\varepsilon} \int_{Q_{\mathbf{k}}^\varepsilon\setminus B(\varepsilon\mathbf{k},r_\varepsilon)} c_{\varepsilon,\mathbf{k}} u_j^{(\varepsilon)} v\,\mathrm{d}\mathbf{x} \to 0. \tag{15}$$

Combining (13), (14) and (15), we have indeed that Condition (**EF2***) is satisfied. □

Proposition 3.6. *For every $j,k\in\mathbb{N}$,*

$$\lim_{\varepsilon\to0}\left|\langle U_j^{(\varepsilon)},U_k^{(\varepsilon)}\rangle_{\mathrm{L}^2(\Omega,\mu)} - \langle u_j^{(\varepsilon)},u_k^{(\varepsilon)}\rangle_{\mathrm{L}^2(\Omega^\varepsilon,\mu^\varepsilon)}\right| = 0.$$

*In other words, Condition (**EF3***) is satisfied.*

Proof. Observe that by the variational characterisation of $\psi_{\mathbf{k}}^\varepsilon$, for all $\varepsilon>0$ small enough and $j,k\in\mathbb{N}$

$$\begin{aligned}\langle U_j^{(\varepsilon)},U_k^{(\varepsilon)}\rangle_{\mathrm{L}^2(\Omega,\mu)} - \langle u_j^{(\varepsilon)},u_k^{(\varepsilon)}\rangle_{\mathrm{L}^2(\Omega^\varepsilon,\mu^\varepsilon)} &= \int_\Omega U_j^{(\varepsilon)}U_k^{(\varepsilon)}\beta\,\mathrm{d}\mathbf{x}\\ &\quad - \sum_{\mathbf{k}\in\tilde{\mathbf{I}}^\varepsilon}\int_{Q_{\mathbf{k}}^\varepsilon\setminus B(\varepsilon\mathbf{k},r_\varepsilon)} \nabla\psi_{\mathbf{k}}^\varepsilon\cdot\nabla(u_j^{(\varepsilon)}u_k^{(\varepsilon)}) + c_{\varepsilon,\mathbf{k}}u_j^{(\varepsilon)}u_k^{(\varepsilon)}\,\mathrm{d}\mathbf{x}.\end{aligned}$$

Again, from the generalised Hölder inequality,

$$\begin{aligned}\sum_{\mathbf{k}\in\tilde{\mathbf{I}}^\varepsilon}\int_{Q_{\mathbf{k}}^\varepsilon\setminus B(\varepsilon\mathbf{k},r_\varepsilon)} \nabla\psi_{\mathbf{k}}^\varepsilon\cdot\nabla(u_j^{(\varepsilon)}u_k^{(\varepsilon)})\,\mathrm{d}\mathbf{x} &\ll \sum_{\mathbf{k}\in\tilde{\mathbf{I}}^\varepsilon}\|\psi_{\mathbf{k}}^\varepsilon\|_{\mathrm{L}^2(Q_{\mathbf{k}}^\varepsilon\setminus B(\varepsilon\mathbf{k},r_{\mathbf{k}}^\varepsilon)}\left(\left\|u_j^{(\varepsilon)}\right\|_{\mathrm{L}^\infty(\Omega)}+\left\|u_k^{(\varepsilon)}\right\|_{\mathrm{L}^\infty(\Omega)}\right)\times\\ &\quad\times\left(\left\|\nabla U_j^{(\varepsilon)}\right\|_{\mathrm{L}^2(Q_{\mathbf{k}}^\varepsilon)}+\left\|\nabla U_k^{(\varepsilon)}\right\|_{\mathrm{L}^2(Q_{\mathbf{k}}^\varepsilon)}\right)\\ &\ll \varepsilon^{d/2}\left(\left\|\nabla U_j^{(\varepsilon)}\right\|_{\mathrm{L}^2(\Omega)}+\left\|\nabla U_k^{(\varepsilon)}\right\|_{\mathrm{L}^2(\Omega)}\right).\end{aligned} \tag{16}$$

Finally, strong $\mathrm{L}^2(\Omega)$ convergence implies that as $\varepsilon\to0$,

$$\int_\Omega U_j^{(\varepsilon)}U_k^{(\varepsilon)}\beta\,\mathrm{d}\mathbf{x} - \sum_{\mathbf{k}\in\tilde{\mathbf{I}}^\varepsilon}\int_{Q_{\mathbf{k}}^\varepsilon\setminus B(\varepsilon\mathbf{k},r_\varepsilon)} \nabla\psi_{\mathbf{k}}^\varepsilon\cdot\nabla(u_j^{(\varepsilon)}u_k^{(\varepsilon)}) + c_{\varepsilon,\mathbf{k}}u_j^{(\varepsilon)}u_k^{(\varepsilon)}\,\mathrm{d}\mathbf{x}\to0. \tag{17}$$

Inserting (16) and (17) yields indeed that Condition (**EF3***) is satisfied. □

We can now have the convergence of the Steklov eigenpairs.

Proof of Proposition 3.3. By construction of Ω^ε and μ^ε, Lemma 3.4 and Propositions 3.5 and 3.6, Conditions (**M1***)–(**M3***) and (**EF1***)–(**EF3***) are satisfied, so that all eigenpairs converge to the corresponding pair. All that is left to prove is the strong convergence of the eigenfunctions. Since we already have weak convergence, we only need to prove convergence of the norms.

It follows from (11) and (12) that

$$\lim_{\varepsilon\to0}\left|\left\|U_j^{(\varepsilon)}\right\|_{\mathscr{L}(\Omega,\mu)} - \left\|u_j^{(\varepsilon)}\right\|_{\mathscr{L}(\Omega^\varepsilon,\mu^\varepsilon)}\right| = 0,$$

and we already have strong $\mathrm{L}^2(\Omega,\mu)$ convergence. Following (8) in Proposition 2.9, the convergence is therefore strong in $\mathscr{L}(M,\mu)$. □

3.3. **Convergence of the Neumann eigenpairs.** We denote this time $\nu_j^{(\varepsilon)}$ the Neumann eigenvalues of Ω^ε, and $v_j^{(\varepsilon)}$ the associated eigenfunctions, and $V_j^{(\varepsilon)} = J^\varepsilon v_j^\varepsilon$. We prove the following.

Proposition 3.7. *As $\varepsilon \to 0$, $\nu_j^{(\varepsilon)} \to \lambda_j(\Omega, g_0, \mathrm{d}\mathbf{x}) = \nu_j(\Omega)$. The harmonic extension $V_j^{(\varepsilon)}$ of the eigenfunctions converges weakly in $\mathrm{H}^1(\Omega)$ to the eigenfunctions associated with $\nu_j(\Omega)$.*

This could be proved by modifying the arguments in [23]. For completeness and consistency, we proceed again by proving that Conditions (**EF1***)–(**EF3***) hold. Note that in this situation it is simpler than for the Steklov eigenfunctions.

Lemma 3.8. *The functions $V_j^{(\varepsilon)}$ are bounded in $\mathrm{L}^2(\Omega)$. In other words, Condition (**EF1**) is satisfied.*

Proof. This follows from the fact that J^ε is a bounded operator, uniformly in ε, and the normalisation $\left\| v_j^\varepsilon \right\|_{\mathrm{L}^2(\Omega^\varepsilon)} = 1$. □

Lemma 3.9. *Let $j \in \mathbb{N}$, and $f \in \mathrm{H}^1(\Omega)$. The limit*

$$\lim_{\varepsilon \to 0} \left| \langle V_j^{(\varepsilon)}, f \rangle_{\mathrm{H}^1(\Omega)} - \langle v_j^{(\varepsilon)}, f \rangle_{\mathrm{H}^1(\Omega^\varepsilon)} \right| = 0$$

*holds. In other words, Condition (**EF2***) is satisfied.*

Proof. Once again, we may choose f in the dense subspace $\mathrm{C}^\infty(\Omega)$. Then, we observe that

$$\left| \langle V_j^{(\varepsilon)}, f \rangle_{\mathrm{H}^1(\Omega)} - \langle v_j^{(\varepsilon)}, f \rangle_{\mathrm{H}^1(\Omega^\varepsilon)} \right| \leqslant \left| \mathbf{T}^\varepsilon \right|^{1/2} \left\| V_j^{(\varepsilon)} \right\|_{\mathrm{H}^1(\Omega)} \left\| f \right\|_{\mathrm{C}^1(\Omega)}$$

which goes to 0 as $\varepsilon \to 0$, from equation (10). □

Proposition 3.10. *For every $j, k \in \mathbb{N}$, the limit*

$$\lim_{\varepsilon \to 0} \left| \langle V_j^{(\varepsilon)}, V_k^{(\varepsilon)} \rangle_{\mathrm{L}^2(\Omega)} - \langle v_j^{(\varepsilon)} v_k^{(\varepsilon)} \rangle_{\mathrm{L}^2(\Omega^\varepsilon)} \right| = 0$$

*holds. In other words, Condition (**EF3***) is satisfied.*

Proof. This follows simply from the fact that as an operator from $\mathrm{H}^1 \to \mathrm{L}^2$, the norm of the harmonic extension operator to a small ball goes to 0 as the radius goes to 0, so that

$$\lim_{\varepsilon \to 0} \int_{\mathbf{T}^\varepsilon} V_j^\varepsilon V_k^\varepsilon \, \mathrm{d}\mathbf{x} = o(1).$$

□

Combining the previous three Lemmas, we have indeed proven Proposition 3.7. We may now prove the main convergence theorem of our section.

Proof of Theorem 3.1. For $j \in \mathbb{N}$ and $\varepsilon > 0$, Proposition 3.2 ensures the existence of $\alpha > 0$ such that

$$\left| \lambda_j(\Omega, g_0, \alpha\iota_* \, \mathrm{d}A_g + \beta \, \mathrm{d}\mathbf{x})(1 + \alpha |\partial\Omega|) - \lambda_j(\Omega, g_0, \beta \, \mathrm{d}\mathbf{x}) \right| < \frac{\varepsilon}{2}.$$

Then, by Propositions 3.3 and 3.7, there is $\Omega^\varepsilon \subset \Omega$ such that

$$\left| \lambda_j(\Omega^\varepsilon, g_0, \mu^\varepsilon) \left|\Omega^\varepsilon\right| - \alpha \lambda_j(\Omega, g_0, \alpha\iota_* \, \mathrm{d}A + \beta \, \mathrm{d}\mathbf{x}) \left(|\partial\Omega| + \alpha^{-1} \int_\Omega \beta \, \mathrm{d}\mathbf{x} \right) \right| < \frac{\varepsilon}{2},$$

and

$$\left|\lambda_j(\Omega^\varepsilon, g_0, \mathrm{d}\mathbf{x}|_{\Omega^\varepsilon}) - \lambda_j(\Omega, g_0, \mathrm{d}\mathbf{x})\right| < \frac{\varepsilon}{2}.$$

Combining those estimates yield exactly Theorem 3.1. □

4. LARGE STEKLOV EIGENVALUES IN THE PLANE

In this section, we prove the isoperimetric inequalities related to the Steklov eigenvalues.

Proof of Theorem 1.1. Let $\Omega \subset \mathbb{R}^2$ be a simply connected Lipschitz domain. For g a smooth metric on the sphere, let

$$\Lambda_j(g) := \lambda_j(\mathbb{S}^2, g, \mathrm{d}v_g)\,\mathrm{Area}_g(\mathbb{S}^2).$$

It is known from [13, 16] that $\sup_g \Lambda_j(g) = 8\pi j$. Let $\delta > 0$, and g be a smooth metric on $\mathbb{S}^2$ such that such that

$$\Lambda_j(g) > 8\pi j - \delta.$$

Let Υ be $\mathbb{S}^2$ with a small disk removed. It is easy to verify that as the radius of that disk goes to 0, the Neumann eigenpairs (μ_j, f_j) of Υ along with $\mathrm{d}v_g$ restricted to Ω satisfy conditions (**M1***)–(**EF3***) with the limit measure being $\mathrm{d}v_g$ so that if the radius of the removed disk is small enough,

$$\nu_j(\Upsilon)\,\mathrm{Area}_g(\Upsilon) > \Lambda_j(g) - \delta.$$

Let $\Phi : \Omega \to \Upsilon$ be a conformal diffeomorphism. Since Dirichlet energy is a conformal invariant, the jth Neumann eigenvalue of Υ is equal to the eigenvalue $\lambda_j(\Omega, g_0, \Phi^*(\mathrm{d}v_g))$. The homogenisation Theorem 3.1 guarantees the existence of $\Omega^\varepsilon \subset \Omega$ such that

$$\sigma_j^{(\varepsilon)}\left|\partial\Omega^\varepsilon\right| > \lambda_j(\mathbb{D}, g_0, \Phi^*(\mathrm{d}v_g)) \int_\Omega \Phi^*(\mathrm{d}v_g) - \delta. \tag{18}$$

Putting this all back together yields the bound $\sigma_j(\Omega^\varepsilon)\,|\partial\Omega^\varepsilon| > 8\pi j - 3\delta$. Since $\delta > 0$ is arbitrary $\Sigma_j^* \geq 8\pi j$. □

Proof of Theorem 1.6. For $\delta > 0$, proceed as in the proof of Theorem 1.1, but start with $\Omega \subset \mathbb{R}^2$ such that $\nu_j(\Omega)\,|\Omega| < \frac{\delta}{2}$, for instance a very thin rectangle. By Theorem 3.1, when choosing ε in (18), ε can be chosen small enough so that $\nu_j(\Omega^\varepsilon)\,|\Omega| < \delta$. This concludes the proof. □

Remark 4.1. It is clear from our constructions that as soon as a measure μ on Ω is a weak-$*$ limit of measures of the form $\beta\,\mathrm{d}\mathbf{x}$ respecting conditions (**M1***)–(**M3***) and (**EF1***)–(**EF3***), we can find Ω^ε such that the normalised Steklov eigenvalues on Ω^ε approximate normalised variational eigenvalues associated with μ on Ω. In particular, following the work of Lamberti–Provenzano [20], if two domains Ω_1, Ω_2 are conformally equivalent, there is a sequence of domains $\Omega^\varepsilon \subset \Omega_1$ such that the normalised Steklov eigenvalues of Ω^ε are arbitrarily close to the normalised Steklov eigenvalues of Ω_2. Furthermore, they can be chosen so that the Hausdorff measure of their respective boundaries converges to 0. By a different construction, we therefore recover a result similar to Theorem 1.4. These domains Ω^ε will be Ω with holes removed, those holes will become bigger as they accumulate near the boundary.

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